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4.5
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DETECTION OF TEMPORALLY COHERENT LIGHT AT LOW SIGNAL LEVELS

Final Report
1 June 1975 - 31 December 1976

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January 1977

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The objective of this study program has been to study methods of detection of coherent or partially coherent optical radiation in a noncoherent background. This study is primarily intended for application at low light levels and low ratios of coherent to noncoherent (i.e., signal/noise) light. This study has application for identifying/locating a source of partially coherent light (e.g., laser) in daylight or in the presence of

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thermal background radiation.

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almost totally noncoherent. Laser radiation is partially
coherent, and can, in certain cases approach total coherence.

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1. INTRODUCTION

The objective of this study program has been to study methods of detection of coherent or partially coherent optical radiation in a noncoherent background. This study is primarily intended for application at low light levels and low ratios of coherent to non-coherent (i.e., signal/noise) light. This study has application for identifying/locating a source of partially coherent light (e.g., laser) in daylight or in the presence of thermal background radiation.

Essentially this measurement method is based upon measurements of temporal coherence of received light. The background light is almost totally noncoherent. Laser radiation is partially coherent, and can, in certain cases approach total coherence.

2. REVIEW OF METHODS FOR MEASURING
TEMPORAL COHERENCE

One of the classical methods of measuring temporal coherence is based upon a form of interferometry. In such methods the beam of light whose coherence is to be measured is separated into two beams. One of the beams is time delayed relative to the other (e.g., by introducing a path length difference), and then the beams are recombined in such a way as to form visible interference fringes. The interference fringe contrast ratio is a function of the temporal coherence which is being sought.

This interferometric measurement could, in principle, be used as a method of detecting laser radiation. However, it tends to be useful for detecting a temporally coherent source only at large signal levels and large signal/noise compared to the intent of our investigation. Thus our study will be devoted more to techniques of measuring temporal coherence which operate at the lowest possible levels.

There are several methods for measuring temporal coherence which are applicable to low signal levels. Each of these involves measurements of statistical parameters of the photon arrival rate. The signal detection method which we have studied in our investigation uses photon arrival rate statistics to estimate various

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moments of the statistical distribution function for the photon arrival rate. We will demonstrate shortly that these are distinctly different for coherent and noncoherent radiation.

In a later section of this report we describe instrumentation and procedures for estimating the moments of the photon arrival rate distribution function from finite numbers of observations. From these we describe a signal processing method which leads to a definition of a signal-to-noise ratio from which the signal detectability can be assessed. We further determine the size of the sample space required for acceptable confidence in the detection of the coherent radiation.

3. BRIEF TUTORIAL REVIEW OF

TEMPORAL COHERENCE

The temporal coherence of an optical field is a measure of its ability to produce stable time independent interference fringes. The temporal coherence can equally well be represented in the time or frequency domain using classical statistical concepts.

In classical statistics the temporal coherence of a scalar stationary random process $V(t)$ is given by the autocorrelation function $R_V(\tau)$ or the power spectral density $W_V(\omega)$

$$R_V(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T V(t) \, V(t + \tau) \, dt$$

$$W_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau) \, e^{-i\omega\tau} \, d\tau$$

In the semi-classical approximation (i.e., ignoring quantum effects) these same functions are useful for representing the temporal coherence of any Cartesian component of an optical electric or magnetic field.

A linearly polarized optical field is conveniently represented by the complex amplitude U which is related to the intensity I by:

$$I = |U|^2 = UU^*$$

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This scalar amplitude is proportional to the electric field peak amplitude E

$$U = \frac{E}{\sqrt{2\eta_0}}$$

where

$$E = ||\bar{E}||$$

$\bar{E}$ = vector electric field

η_0 = wave impedance

In the above both U and E are complex analytic functions which are derived from the corresponding real electric fields.

The coherence of an optical field is represented by the mutual coherence function $\Gamma(\bar{r}_1, \bar{r}_2, \tau)$ where:

$$\Gamma(\bar{r}_1, \bar{r}_2, \tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} U^*(\bar{r}_1, t) \, U(\bar{r}_2, t + \tau) \, dt$$

where

$$\bar{r}_{1,2} = \text{vector positions}$$

$$\tau = \text{time difference}$$

It is convenient to denote $\Gamma(\bar{r}_1, \bar{r}_2, \tau)$ by the symbol $\Gamma_{12}(\tau)$. In this entire discussion we are assuming that U is ergodic.

From this definition of the mutual coherence function the corresponding frequency domain function $G(\bar{r}_1, \bar{r}_2, \nu)$ can be found:

$$G(\bar{r}_1, \bar{r}_2, \nu) = \int_{-\infty}^{\infty} \Gamma(\bar{r}_1 \bar{r}_2 \tau) e^{i2\pi\nu\tau} \, d\tau$$

This function is called the mutual spectral density or sometimes the cross spectral density.

In most calculations it is helpful to use normalized functions. The normalized mutual coherence function is called the degree of coherence and is denoted

$$\gamma(\bar{r}_1\bar{r}_2\tau) = \gamma_{12}(\tau)$$

$$\gamma_{12}(\tau) = \frac{\Gamma(\bar{r}_1\bar{r}_2\tau)}{\{\Gamma(\bar{r}_1\bar{r}_1 0) \ \Gamma\bar{r}_2\bar{r}_2 0\}^{1/2}}$$

Note that $\gamma_{12}(\tau)$ is bounded:

$$|\gamma_{12}(\tau)| \leq 1$$

The corresponding normalized spectral density $g_{12}(\nu)$ is given by

$$g_{12}(\nu) = \int_{-\infty}^{\infty} \gamma_{12}(\tau) e^{-i2\Pi\nu\tau} \, d\tau$$

$$= \frac{G_{12}(\nu)}{\int_0^{\infty} G_{12}(\nu) \, d\nu}$$

There is an important special case when considering light from a laser. Laser generated light occupies a very narrow spectral width $\Delta\nu$. In this case the normalized spectral density can be approximated by:

$$g(\nu) = g \qquad \langle\nu\rangle - \frac{\Delta\nu}{2} \leq \nu \leq \langle\nu\rangle + \frac{\Delta\nu}{2}$$

where

$\langle\nu\rangle$ = average frequency

$\Delta\nu$ = linewidth

$\frac{\Delta\nu}{\langle\nu\rangle} \ll 1$

In this case the normalized degree of coherence is given by

$$\gamma_{12}(\tau) = \gamma_{12}(0)\, \gamma_{11}(\tau)$$

where

$$\gamma_{11}(\tau) = \text{sinc}\, \frac{\Delta\nu\tau}{2}$$

$$\gamma_{12}(0) = \frac{1}{\left[I_1\, I_2\right]^{1/2}} \int_{\Sigma} I_s \frac{e^{-i\,\frac{2\Pi\langle\nu\rangle}{c}\,(r_1 - r_2)}}{r_1 r_2}\, ds$$

and where

I_s = source intensity

I_1 = intensity at vector position $\bar{r}_1$

I_2 = " " " " $\bar{r}_2$

$r_{1,2}$ = distance from differential source position $d\tau$ to points 1,2 respectively

ds = differential element of area at light source

Σ = entire source area which is
radiating light

This particularly important special case is known
as the Van-Cittert - Zernike theorem. It has the importance
of separating the spatial coherence $\gamma_{12}(0)$ and the
temporal coherence which is given by $\gamma_{11}(\tau)$. In the
remainder of our work we will make the simplifying
assumption that the light whose temporal coherence we
are attempting to measure occupies a very narrow spectrum
$\Delta\nu$ compared to the average frequency $<\nu>$. Thus in
this study we are attempting to measure $\gamma_{11}(\tau)$ by
various methods rather than attempting to find $\gamma_{12}(z)$.

The method which we have been studying for the
measurement of $\gamma_{11}(\tau)$ is based upon measurements of
the statistics of photon arrival rates. This study
has been motivated by the work in Ref. 4 which showed
the relationship between $\gamma_{11}(\tau)$ and photon statistical
measurements. There it is shown that for light having
Gaussian amplitude statics we have:

$$p_c(\tau) = \eta < I(t) > \left[1 + |\gamma_{11}(\tau)|^2\right]$$

where

$p_c(\tau)$ = probability of receiving a
photon at time $t + \tau$ given
a photon at time t

$$\langle I(t) \rangle \;=\; \text{average light intensity}$$

$$\eta \;=\; \text{detector efficiency}$$

$$\langle n \rangle \;=\; \text{average number of photons}$$
$$\text{generated/unit time by } \langle I(t) \rangle$$

This formulation is not strictly applicable to the detection of laser light because the latter does not have Gaussian amplitude statistics. However it is shown (Ref. 4) that noncoherently generated photons have a tendency to bunch together having a nonuniform arrival rate; whereas perfectly coherent constant amplitude light has a uniform arrival rate. This result suggests that the arrival rate statistics are different for coherently generated photons that for noncoherently generated photons. A significant portion of the research conducted during this study program has been devoted to a careful investigation of the statistics of photon arrival rates.

Section 5 of this report describes the results of our theoretical study of photon arrival rate statistics. There it is shown that certain of the statistical moments for the photon arrival rate are sufficiently different for coherent and noncoherent light to provide the basis of detection of the former when concealed by the latter. In Section 6 we define a signal quantum from which it is possible to detect coherent light in a noncoherent background.

4. INSTRUMENTATION FOR OPTICAL SIGNAL

DETECTION METHOD

The method which we are studying for detection
of coherent radiation can be understood with reference
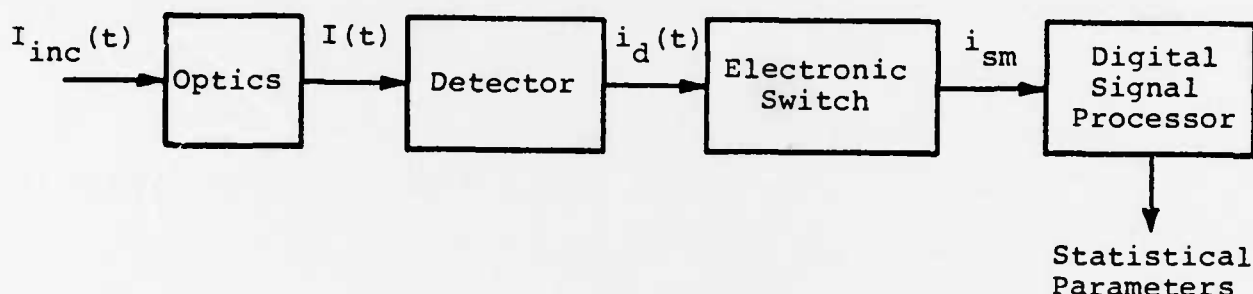to Fig. 1.

Figure 1

This figure is a block diagram of a portion of the
optical signal detection instrumentation. This
instrument, which is presumed to be receiving light from
some region of space, receives light having intensity
I_{inc} from that space. This region of space is that
portion contained inside a solid angle which is deter-
mined by the instrument optics.

The optics include a lens system which might,
for example, be a telescope and possibly some sort of
preselector or filter such as a Fabry Perot etalon.
The parameters of these optical components are a function
of the intended application of the system. For example,
in attempting to detect the radiation from a particular
type of laser at one of its known wavelengths, it would

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be reasonable to remove the light outside of the inhomogeneously broadened linewidth for that particular laser and wavelength by means of an optical filter.

Referring to the previous section we assumed that the spectral width of the optical source $\Delta \nu$ is small compared to the mean frequency $<\nu>$. The preselector filter which we mentioned above is a means of achieving this narrow spectral profile.

Other factors which influence the specification of the parameters of the system optics are: the desired region of coverage, the total number of wavelengths which must be covered and the corresponding linewidths/bandwidths. In this study we presume that the incident light is filtered about a center frequency ν_a where ν_a is the average optical frequency over the filter passband (i.e., $\nu_a = <\nu>$).

The light which emerges from the system optics reaches the active area of a photo detector. In our study we are considering low light levels and low signal/noise. A detector of great sensitivity will be required for this purpose which in limiting cases should be capable of detecting individual photons. That is, the detector output signal must rise above the quiescent noise level in the presence of a single photon.

Moreover we will be concerned with rapid variations in the intensity of the light. The detector should be capable of responding to variations in light intensity

which occur in intervals on the order of nanoseconds. This implies a detector having a bandwidth of the order of a few GHz. The combination of large effective detector bandwidth and sensitivity suggests that the detector will probably have to be a cooled photo multiplier.

However, it is not the purpose of this investigation to develop a design for an instrument. Therefore we will not pursue the characteristics of the detector further than to say that it must be capable of detecting very small numbers of photons and must have a bandwidth in excess of about 10^9 Hz.

In the remainder of this report it will be assumed that the detector is a photo multiplier having a multiplication factor M_{pm}. That is the ratio of anode current i_d to cathode current $i_c(t)$ is given by:

$$\frac{i_d}{i_c} = M_{pm}$$

The cathode current is determined by the number of incident photons, by the quantum efficiency of the cathode and by the optical frequency. Assuming that the average energy per photon in the filtered light reaching the cathode (i.e., $h\nu_a$) is sufficient to overcome the cathode work function, then some nonzero fraction of the incident photons will generate free electrons at the cathode. Denoting the number of incident photons/ unit time which strike the cathode N_p and the number of liberated electrons/unit time N_e, then

$$N_e = \zeta N_p$$

where ζ is the cathode quantum efficiency. It is convenient to call N_p the photon flux rate.

This photon flux rate is linearly proportional to intensity $I(t)$:

$$N_p = \frac{1}{h\nu_a} \int_{A_d} I\ ds$$

where A_d is the active area of the detector which received intensity I. In writing this expression we assume normal incidence upon a planar cathode. For our analysis we make the simplifying assumption that I is uniform over A_d for which we obtain:

$$N_p = \frac{I\ A_d}{h\nu_a}$$

The cathode current is i_c given by

$$i_c = q\ N_e$$

and the anode current is given by:

$$i_a = q\ N_e\ M_{pm}$$

In writing these expressions we have ignored the transit time of electrons. Ignoring these effects is equivalent to assuming that the detector has infinite detection bandwidth. We will make this assumption throughout this report.

Our signal detection method is based upon the statistics of the photon flux rate. In collecting data for the required statistical calculations it is necessary to sample the detector output current. This is accomplished by means of a very fast electronic switch. Denoting the switched output current $i_{s,m}(t)$ for the mth sample we can write

$$i_{s,m}(t) = i_d(t) \qquad t_m \leq t < t_m + T$$

$$= 0 \qquad t_m + T \leq t < t_{m+1}$$

$$t_m = m\tau$$

$$m = 1, 2, \ldots, M$$

$$\tau = \text{sampling period}$$

$$T = \text{sampling interval}$$

$$\tau > T$$

$$M = \text{number of samples taken}$$

It will be shown presently that there is a difference in the statistics of total number photo-electrons which are generated in an interval T for coherent and noncoherent light. We define, therefore, the integrated sampled photon arrival rate n_m:

$$n_m = \int_{t_m}^{t_m+T} N_e(t) \, dt$$

$$= \frac{1}{qM_{pm}} \int_{t_m}^{t_m+T} i_a(t)dt = \frac{1}{qM_{pm}} \int_{t_m}^{t_m+T} i_{sm} \, dt$$

We can relate n_m to the intensity $I(t)$ by using the relationships presented above:

$$n_m = \frac{\zeta A_d}{h\nu_a} \int_{t_m}^{t_m+T} I(t) \, dt$$

$$= \eta \int_{t_m}^{t_m+T} I(t) \, dt$$

where

$$\eta = \frac{\zeta A_d}{h\nu_a}$$

The instrumentation depicted in Fig. 1 is assumed to include a digital processor. This processor will collect data (n_m) from M samples and compute various statistical parameters on this sample space as indicated below.

5. THEORY OF OPTICAL SIGNAL DETECTION METHOD

For an understanding of the present optical detection method it is helpful to examine the relationship between the statistical distribution function for the photon arrival rate and the optical intensity $I(t)$. For our study we need the probability $p(n, T)$ (i.e., ergodic statistics) of emitting n photo electrons in time interval from t to $t + T$ where T is fixed. It is shown in Ref. 1 that for perfectly coherent light of intensity $I(t)$ we have:

$$p(n,T,t) = \frac{1}{n!}\left[\eta \int_t^{t+T} I(t')dt'\right]^n \exp\left[-\eta \int_t^{t+T} I(t')dt'\right]$$

Note that this reduces to the Poisson distribution if the mean number of emitted photo electrons is denoted $\langle n \rangle$:

$$p(n,T) = \frac{1}{n!}(\eta W)^n e^{-\eta W}$$

where

$$\langle n \rangle = \eta W$$

$$W = \int_t^{t+T} I(t')dt'$$

$$\eta = \frac{\zeta A_d}{h\langle \nu \rangle}$$

$$\zeta = \text{quantum efficiency of detector}$$

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$$\langle \nu \rangle \;=\; \text{average optical frequency}$$

$$h \;=\; \text{Planks constant}$$

For noncoherent light $I(t')$ is a stochastic process and the photon counting distribution must be averaged over all possible $I(t)$.

$$p(n,\, T) \;=\; \frac{1}{n!} \langle (\eta W)^n \exp(-\eta W) \rangle$$

$$=\; \frac{1}{n!} \int_0^\infty (\eta W)^n \exp(-\eta W)\, P(W)\, dW$$

We can illustrate the use of this relationship by finding $p(n,\, T)$ for an ideal amplitude stabilized laser. In such a case the probability density function $p(I)$ for the intensity is given by:

$$p(I) \;=\; \delta(I - \langle I \rangle)$$

where

$$\langle I \rangle \;=\; \text{average intensity}$$

and where $\delta(\cdot)$ is the Dirac delta function. The integrated intensity W is given by

$$W \;=\; IT$$

The photo detection equation becomes:

$$p(n,\, T) \;=\; \frac{1}{n!} \langle \eta W \rangle^n\, e^{-\langle \eta W \rangle}$$

$$= \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle}$$

where $\langle n \rangle = \eta \langle IT \rangle = \eta \langle I \rangle T$ is the average number of photo electrons generating in an interval of duration T. Note that this is the Poisson distribution.

Another important special case merits attention. We consider the photon counting distribution for a thermal source. An optical field, which is generated by a thermal source composed of many independent atomic radiators, consists of many different frequency components occupying a fixed band. From the central limit theorem it can be concluded that the resulting optical field corresponds to a zero mean Gaussian random process in scalar amplitude.

Assuming a linearly polarized thermally generated optical field then the complex scalar amplitude can be denoted U where:

$$U = U_r + i U_i$$

and where

$$U_r = \text{Re } U$$

$$U_i = \text{Im } U$$

The joint probability density function for these components are given by:

$$p(U_r , U_i) = \frac{1}{2\pi\sigma^2} \exp -\left(\frac{U_r^2 + U_i^2}{2\sigma^2} \right)$$

where σ^2 is the variance in the intensity I:

$$I = |U|^2$$

$$\sigma^2 = \text{Var}(|U|^2) = \frac{<I>}{2}$$

We can write this joint density function in polar form where

$$U = |U|e^{i\phi}$$

$$|U| = \sqrt{U_r^2 + U_i^2}$$

$$\tan \phi = \frac{U_i}{U_r}$$

Letting $|U| = \sqrt{I}$ we obtain

$$P(\sqrt{I}, \phi) = \frac{\sqrt{I}}{2\pi\sigma^2} \exp\left(-\left(\frac{(\sqrt{I})^2}{2\sigma^2}\right)\right)$$

$$= \frac{\sqrt{I}}{\pi<I>} e^{-\frac{(\sqrt{I})^2}{<I>}}$$

This probability density function is independent of the phase ϕ of the scalar amplitude. Thus we can integrate the joint density function over ϕ to obtain the probability density function for the intensity alone:

$$p(\sqrt{I}) = \int_0^{2\pi} p(\sqrt{I}, \phi)d\phi$$

$$= \frac{2\sqrt{I}}{<I>} e^{-\frac{I}{<I>}}$$

This function can be further converted from a function of $\sqrt{I}$ to a function of I by noting

$$dI = 2 \sqrt{I} \, d \sqrt{I}$$

Thus we obtain

$$p(I) = \frac{1}{\langle I \rangle} e^{-\frac{I}{\langle I \rangle}}$$

In our instrumentation we will be using very fast detectors whose outputs will be sampled for very short intervals compared to the time in which $I(t)$ changes appreciably. In this case letting T be the sampling interval we can approximate the integral for W as follows

$$W = \int_t^{t+T} I(t') \, dt'$$

$$\approx IT$$

From this we obtain for $p(n, T)$:

$$p(n, T) = \int_0^\infty \frac{(\eta IT)^n}{n!} \exp(-\eta IT) \frac{e^{-\frac{I}{\langle I \rangle}}}{\langle I \rangle} \, dI$$

$$= \frac{\langle n \rangle^n}{1 + \langle n \rangle^{1+n}}$$

where $\langle n \rangle = \eta \langle I \rangle T$ is equal to the average number of

photo electrons generated in the interval T by the
thermally produced intensity I.

Observe that p(n, T) is different for the two
special cases which we have considered. Moreover the
functions p(n, T) for a practical case is also
different from that for thermally generated light.
It is the difference in p(n, T) which serves as the
basis for our method of detecting laser radiation.

To enhance this notion it is instructive to consider
the mean and variance for these two distributions. If
we imagine that the light which is incident upon our
detection apparatus is totally coherent then we can
denote the optical intensity $I_c(t)$ and the number of
photo electrons generated in the sampling interval T.
We further presume that this light is generated in a
perfect amplitude stabilized laser such that the
Poisson arrival rate statistics apply. Denoting the
mean photon arrival rate $\langle n_c \rangle$ the variance is given
by

$$V(n_c) = \sum_{n=1}^{\infty} \frac{(n_c - \langle n_c \rangle)^2 \, e^{-\langle n_c \rangle}}{n_c!} \langle n_i \rangle^{n_c}$$

$$= \langle n_c \rangle$$

This result is a well known property of the Poisson
process.

On the other hand, assuming that the incident light is totally noncoherent we denote the intensity I_i and the numbers n_i of photo electrons generated in interval T. Denoting the average of this number of photo-electron $\langle n_i \rangle$ we obtain for the variance:

$$V(n_i) = \sum_{n_i=1}^{\infty} \frac{(n_i - \langle n_i \rangle)^2 \langle n_i \rangle^{n_i}}{(1 + \langle n_i \rangle)^{1+n_i}}$$

$$= \langle n_i \rangle (1 + \langle n_i \rangle)$$

Note that if we take the ratio of variance to mean for the two cases we have the basis for discrimination between coherent and noncoherent radiation. We denote this ratio S_c and S_i for coherent and noncoherent respectively and obtain:

$$S_c = \frac{V(n_c)}{\langle n_i \rangle} - 1 = 0$$

$$S_i = \frac{V(n_i)}{\langle n_i \rangle} = 1 + \langle n_i \rangle - 1 = \langle n_i \rangle$$

These functions are sketched in Fig. 2. By measuring the mean and variance of the detected photo electrons we can, in principle, distinguish perfectly coherent from perfectly noncoherent light. Note that it is advantageous to pick T large enough such that

6. FORMULATION OF SIGNAL DETECTION PROBLEM

In the previous section of this report it has been shown that the statistics of the photon arrival rate are a function of temporal coherence. This influence of coherence is expressed by the probability density function for the number of photons n counted by a photo detector:

$$p(n) = \frac{1}{(n + m - 1)!} \left(1 + \frac{m}{\langle n_i \rangle}\right)^{-n} \left(1 + \frac{\langle n_i \rangle}{m}\right)^{-m} \exp\left[-\frac{\langle n_c \rangle m}{\langle n_i \rangle \rangle m}\right] \times$$

$$L_n^{m-1} \left[-\frac{\langle n_c \rangle m^2}{\langle n_i \rangle (\langle n_i \rangle + m)}\right]$$

where

$$L_n^{m-1} (\ ) = \text{the Laguerre polynomial}$$

$$m = \text{number of modes in the optical field}$$

$$\langle n_i \rangle = \text{average number of noncoherent photons counted in an interval } T$$

$$\langle n_c \rangle = \text{average number of coherent photons counted in } T$$

The number of modes m of the optical field is determined by the receiver optics. It is shown in Ref. 1 that this number of modes is given by:

$$m = \frac{SS^1}{R^2 \lambda^2} \frac{T}{\tau}$$

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where

S = detector area

S^1 = source area

R = distance from sound to detector

λ = optical wavelength

T = sampling interval in which
photon rate is

τ = coherence time of source

In the proposed instrumentation, the detector output will be sampled for a time T. The number of photons in the mth sampling interval of duration T is denoted n_m. If this sampling is taken near the nyquist limit, then T will be of the order of the reciprocal of the detector bandwidth β_d:

$$T \simeq \frac{1}{\beta_d}$$

A typical value for T is in the range from 10^{-6} to 10^{-9}.

In most applications the wavelength of the laser being sought is known. In this case it is reasonable to utilize an optical preselector to filter the light reaching the detector. A Fabry Perot etalon is an example of such a preselector.

Normally it is desirable to have the preselector bandwidth as narrow as possible for maximum pre-signal processing signal/noise. A Fabry Perot etalon is commercially

available having a bandwidth $\Delta\lambda$ of about 1Å. The light emerging from a filter of this bandwidth will have a "coherence time" τ of the order of

$$\tau \simeq \frac{\lambda^2}{C\Delta\lambda}$$

where C equals speed of light and λ wavelength. A 1Å filter resonant at 1 μm will yield a coherence time of about 10^{-10} sec.

It is instructive to estimate the number of modes in a filtered optical field for a typical set of optics that could be used for detecting coherent light in a noncoherent background. The following "reasonable" values are taken for the parameters in the expression for m:

$$S \sim 1 \text{ cm}^2$$
$$S^1 \sim 1 \text{ m}^2$$
$$R \sim 1 \text{ Km}$$
$$\lambda \sim 1 \text{ μm}$$

For this representative example the number of modes is given by

$$m \simeq 100$$

In general the number of modes in an optical field appropriate for the proposed signal detection scheme will be in the range $100 \leq m \leq 10^4$. However, at times, single mode fields

will be considered in the following analysis for simplicity
in interpreting certain results.

The objective of this investigation has been to find
means of detecting coherent photons in the presence of
noncoherent photons. This is accomplished by measuring
various moments of the distribution $p(n)$ and by knowing
the influence of nonzero $\langle n_i \rangle$ upon these moments. Earlier
we showed that the variance of $p(n)$ depends upon $\langle n_i \rangle$
and $\langle n_c \rangle$. In doing so we demonstrated the variance for
the two special cases; (a) $\langle n_i \rangle = 0$ and $\langle n_c \rangle \neq 0$ and
(b) $\langle n_c \rangle = 0$ and $\langle n_i \rangle \neq 0$.

It is instructive to consider the variance $V(n)$ for
the above density function in the case of combined coherent
and noncoherent light because this is the applicable case
for the detection of coherent light in a noncoherent back-
ground.

The variance of $p(n)$ is most easily computed from
the recurrence relation for the moments $\langle n^k \rangle$. In Appendix
A it is shown that

$$\langle n^{k+1} \rangle = \langle n_c + n_i \rangle \langle n^k \rangle + \langle n_i \rangle \left(1 + \frac{\langle n_i \rangle}{m} \right) \frac{\partial \langle n^k \rangle}{\partial \langle n_i \rangle}$$

$$+ \langle n_c \rangle \left(1 + \frac{2 \langle n_i \rangle}{m} \right) \frac{\partial \langle n^k \rangle}{\partial \langle n_i \rangle}$$

where m is the number of modes of the optical field
being investigated. The coherent and noncoherent photons

are independent because they are generated by independence physical mechanisms. As a result the first moment of $p(n)$ is given by

$$\langle n \rangle = \langle n_c + n_i \rangle = \langle n_c \rangle + \langle n_i \rangle$$

From the recurrence relation the second moment is found to be:

$$\langle n^2 \rangle = \left(\langle n_c \rangle + \langle n_i \rangle \right)^2 + \langle n_i \rangle \left(1 + \frac{\langle n_i \rangle}{m} \right) + \langle n_c \rangle \left(1 + \frac{2 \langle n \rangle}{m} \right)$$

The variance [i.e., $V(n)$] in $p(n)$ is given by

$$
\begin{aligned}
V(n) &= \langle (n - \langle n \rangle)^2 \rangle \\
&\doteq \langle n^2 \rangle - \langle n \rangle^2 \\
&= \left(\langle n_c \rangle + \langle n_i \rangle \right)^2 + \langle n_i \rangle \left(1 + \frac{\langle n_i \rangle}{m} \right) \\
&\quad + \langle n_c \rangle \left(1 + \frac{2 \langle n_i \rangle}{m} \right) - \left(\langle n_c \rangle + \langle n_i \rangle \right)^2 \\
&= \langle n_c \rangle \left(1 + \frac{2 \langle n_i \rangle}{m} \right) + \langle n_i \rangle \left(1 + \frac{\langle n_i \rangle}{m} \right)
\end{aligned}
$$

It is convenient to define the ratio of average coherent to noncoherent photons

$$r = \frac{\langle n_c \rangle}{\langle n_i \rangle}$$

Using this notation the variance $V(n)$ becomes:

$$V(n) = \langle n \rangle \left\{ \frac{r}{1+r} \left[1 + \frac{2 \langle n \rangle}{m(1+r)} \right] + \frac{1}{1+r} \left[1 + \frac{\langle n \rangle}{m(1+r)} \right] \right\}$$

Note that this variance which depends significantly upon r can serve as the basis for detecting coherent light. Totally noncoherent light corresponds to the special case r = 0. Totally coherent light corresponds to the limit as r tends to infinity. For any given signal/noise (i.e., for any given r) the detection of coherent light involves distinguishing between V(n,r) and V(n,o). It is the purpose of the following analysis to consider the process.

6.1 Definition of Signal

In attempting to detect temporally coherent light in a noncoherent background from the photon arrival statistics, the received signal must be sampled. In the proposed scheme, the number of photons is counted for an interval T repeatedly. Denoting the number of photons received in the mth sample M_m, then an estimate of the variance $\hat{V}$ of n can be obtained

$$\hat{V}(n) \;=\; \frac{1}{N-1} \sum_{m=1}^{N} (n_m - <\hat{n}>)^2$$

where

$$<\hat{n}> \;=\; \frac{1}{N} \sum_{m=1}^{N} n_m$$

i.e., $<\hat{n}>$ is an estimate of the mean number of photons in an interval T and where N is the number of samples taken.

For mathematical convenience we assume that N is sufficiently large that the estimate of <n> is essentially an accurate measure of <n>:

$$\hat{<n>} \simeq <n> = \lim_{N\to\infty} \hat{<n>}$$

It is inadequate to consider the variance of n for our purposes. However the relationship between variance and mean of n can serve as the basis for detection of coherent light.

In Section 5 of this report we introduced a definition of signal:

$$S = \frac{V(n)}{<n>} - 1$$

We use a similar definition here for S, which can be written in the following form:

$$S = \frac{V(n)}{<n>} - 1$$

$$= \frac{<n>}{m} \frac{1 + 2r}{(1+r)^2}$$

It is important to recognize that the proposed instrumentation scheme for detecting coherent light is capable of only estimating S. That is denoting this estimate $\hat{S}$, the proposed instrumentation can perform computations leading to a value for $\hat{S}$

$$\hat{S} = \frac{\hat{V}(n)}{\hat{<n>}} - 1$$

The quantity denoted S, which we have called signal, is the expected value of this estimate:

$$S = <\hat{S}> = \lim_{N \to \infty} S$$

6.2 Post Signal Processing Noise

For any finite sample space (i.e., finite N) the estimate of $\hat{S}$ will be in error by an amount denoted ε

$$\varepsilon = \hat{S} - S$$

where ε is a random variable. A convenient measure of the size of the error ε is the variance in $\hat{S}$, i.e.,

$$
\begin{aligned}
V(\hat{S}) &= <(\hat{S} - S)^2> \\
&= <\varepsilon^2> \\
&= <\left[\frac{V(n)}{\hat{<n>}} - 1 - \frac{V(n)}{<n>} + 1\right]^2> \\
&= <\left[\frac{\hat{V}(n)}{\hat{<n>}} - \frac{V(n)}{<n>}\right]^2>
\end{aligned}
$$

In principle we could expand the above expression in terms of the variance of $\hat{V}$ and $\hat{<n>}$. However, the computations are unwieldy and will yield results that are not readily interpreted. Fortunately for relatively large

sample space (i.e., large N) the variance $\langle \hat{n} \rangle$ is small compared with the variance in $\hat{V}$:

$$V\left[\langle \hat{n} \rangle\right] = \frac{V(n)}{N} \ll V\left[\hat{V}(n)\right]$$

The validity of the above inequality can be established by numerical example. In this case the following approximation can be made:

$$\langle \hat{n} \rangle \approx \langle n \rangle$$

Substituting this approximation into the expression for $V(\hat{S})$ yields

$$V(\hat{S}) \approx \frac{1}{\langle n \rangle^2} V\left[\hat{V}(n)\right]$$

It is shown in Ref. 3 that $V\left[\hat{V}(n)\right]$ is given approximately by

$$V\left[\hat{V}(n)\right] \approx \frac{\mu_4 - 2\mu_2^2}{N}$$

where μ_4 and μ_2 are the fourth and second central moments of $p(n)$ respectively:

$$\mu_4 = \langle (n - \langle n \rangle)^4 \rangle$$

$$= \sum_{n=1}^{\infty} (n - \langle n \rangle)^4 \, p(n)$$

$$\mu_2 = \langle (n - \langle n \rangle)^2 \rangle$$

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$$\mu_2 = \sum_{n=1}^{\infty} (n - <n>)^2 \, p(n)$$

It is possible to evaluate μ_4 and μ_2 from the moments $<n^k>$ ($k = 1, 2, 3, 4$) using the recurrence relationship that was developed in Appendix A. In Appendix B it is shown that the variance in the signal is given by:

$$V(\hat{S}) = \frac{1}{N<n>^2} \left\{ 6 \, \frac{r<n>^2}{(1+r)^2} \left(1 + \frac{<n>}{m(1+r)}\right)\left(1 + \frac{2<n>}{m(1+r)}\right) \right.$$

$$+ \left[\frac{r<n>}{1+r}\left(1 + \frac{2<n>}{m(1+r)}\right) + \frac{<n>}{1+r}\left(1 + \frac{<n>}{m(1+r)}\right) \right]$$

$$\left[\frac{r<n>}{1+r}\left(1 + \frac{2<n>}{m(1+r)}\right) + \frac{3<n>}{m(1+r)}\left(1 + \frac{<n>}{m(1+r)}\right) \right.$$

$$\left. \left. + \left(1 + \frac{2<n>}{m(1+r)}\right)^2 \right] \right\}$$

In characterizing measurement errors it is common practice to specify the standard deviation σ (i.e., square root of variance):

$$\sigma(\hat{S}) = \sqrt{V(\hat{S})}$$

If the proposed signal detection method were implemented with ideal instrumentation (i.e., zero instrument noise), then the noise associated with this scheme results from the finite sample space from which $\hat{S}$ is estimated. A convenient measure of this noise is $\sigma(\hat{S})$ defined above.

$$N_o = \sigma(\hat{S})$$

It is shown above that S is given by

$$S = \frac{<n>}{m} \frac{1 + 2r}{(1+r)^2}$$

Figure 3 is a sketch of signal S vs <n>. Note that
S(<n>) is a linear function of <n> having a slope
$\frac{1 + 2r}{M(1+r)^2}$ for any given ratio r and *number of modes* m.
The essential features of our signal detection
problem can be understood with respect to Fig. 3. The
instrumentation must distinguish between a signal having
r identically zero (i.e., no coherent light) and light
having small but nonzero r. The slope of the S(<n>)
curves is a decreasing function of r having the maximum
value at r = 0 (i.e., zero signal) and having zero slope
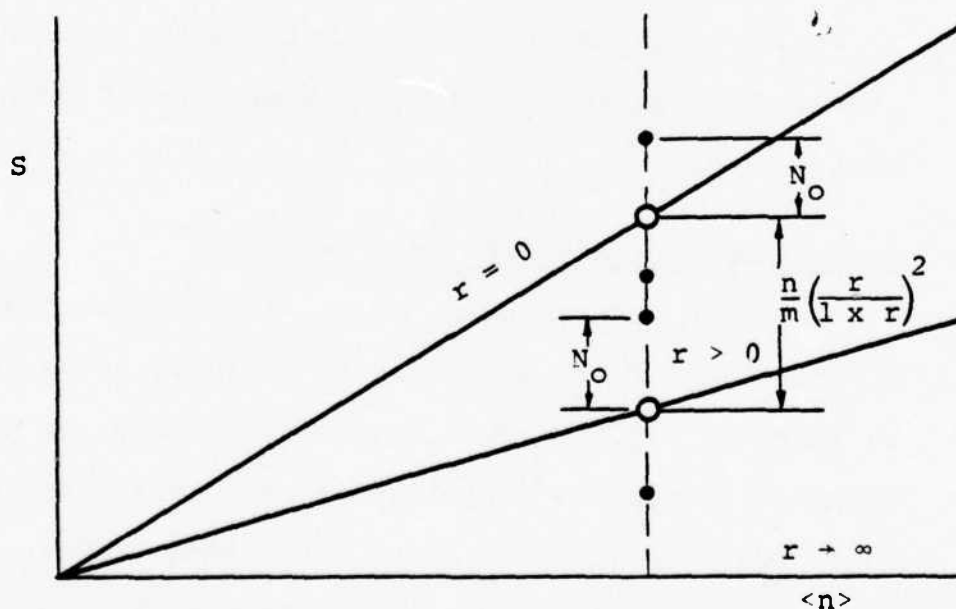for r→∞ (i.e., totally coherent optical field).

Figure 3

The signal detection problem involves a measurement of r in order to determine nonzero r. If it weren't for noise, the signal detection problem would be straight-forward. A suitable signal detection instrument would measure $<n>$ and then $S<n>$ for the given (and fixed) m could be computed. From this r could be determined by inverting the expression for $S(<n>, r)$.

Unfortunately there is considerable noise associated with the proposed detection scheme, which places a lower limit upon detectable r. As mentioned above, there is optical receiver noise, which, for convenience, is being neglected. The fundamentally unavoidable noise associated with the proposed signal detection method results from the finite sample estimate of $\hat{S}$ as explained above. This noise explains the uncertainty in the measurement of S thereby placing a limit on the minimum r which can be measured.

6.3 Example Signal Detection Method

For an understanding of the limits placed upon the minimum size of r which can be measured it is helpful to refer to Fig. 4. This Fig. 4 is a sketch of the probability density function for $\hat{S}$. This function tends to express the distinction of measurement results for $\hat{S}(r)$ for any given true $S(r)$. Here we have depicted the conditional probability for $\hat{S}$ under the two
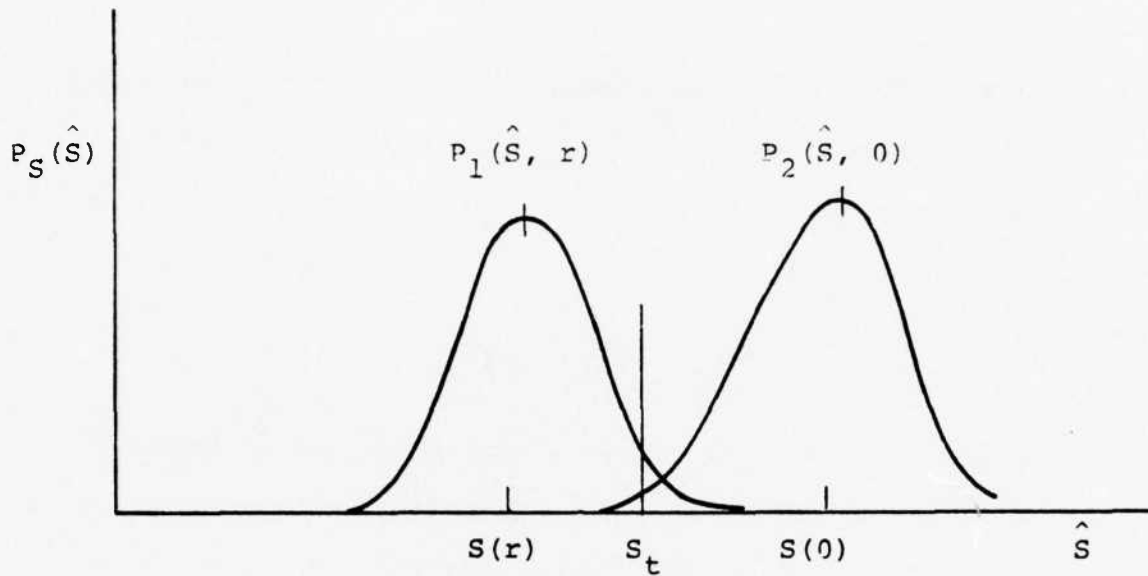
Figure 4

hypotheses $r = 0$ and $r > 0$. The variance for these two conditional density functions are $V[\hat{S}(0)]$ and $V[\hat{S}(r)]$ respectively. Recall that we have shown these variances to be inversely proportional to the size of the sample space N.

6.4 Feasibility of Proposed Methods

In the following subsection of this report the feasibility of this proposed signal detection method is considered with respect to certain practical limitations. In considering this feasibility, it is helpful to consider a specific example signal detection method.

In one very traditional implementation of a signal detection scheme a simple threshold decision is reached. The decision will be reached that signal is present whenever

$$r \geq r_t$$

where r_t is the threshold signal-to-noise ratio for reaching the decision that the signal is present. Referring to Fig. 4 this criterion for signal present is equivalent to the condition

$$S(r) \leq S_t$$

where $S_t = S(r_r)$. These conditions are equivalent because S is a monotonic nonincreasing function of r.

In the proposed implementation both $\langle \hat{n} \rangle$ and $\hat{S}$ would be obtained from N samples $n_m (m = 1, 2, ..., N)$. In our above analysis it has been assumed that N is sufficiently large that $\langle \hat{n} \rangle \approx \langle n \rangle$. With known $\langle n \rangle$ and m (i.e., m fixed by configuration), it is possible to compute $S(0)$

$$S(0) = \frac{\langle n \rangle}{m}$$

The minimum desirable detectable signal level r_{min} is selected, and $S_{min} = S(r_{min})$ is then determined. The threshold level S_t will be somewhere between these two values of S.

$$S_{min} < S_t < S(0)$$

depending upon the choice of acceptable errors.

As in all such signal detection methods there are two classes of error which can be made: (1) failure to

detect the signal when it is present and, (2) deciding
the signal is present when there is none. The probability
of an error of the first (i.e., P_{e1}) is given by

$$P_{e1} = \int_{S_t}^{\infty} P_1(\hat{S})dS$$

and the probability of the second type is (i.e., P_{e2}):

$$P_{e2} = \int_0^{S_t} P_2(\hat{S})d\hat{S}$$

The relative size of these two error probabilities depends
upon the function $p_1(S)$ and $p_2(S)$ and upon the choice
of S_t. Selection of an optimum value for S_t is well
covered in the general theory of signal detectability and
is not considered here.

Although this investigation is not a study of signal
detection theory, we can consider a simplified and interes-
ting special case for illustrative purposes. We assume that
p_1 and p_2 have identical shapes (i.e., identical central
moments) but different mean values [i.e., $S(r) \neq S(0)$].
We further assume for mathematical convenience that it is
desirable for $P_{e1} = P_{e2}$. If this is the criterion for
errors, then S_t should be halfway between S_{min} and
$S(0)$:

$$S_t = \frac{S(0) + S_{min}}{2}$$

Furthermore the total probability of error p_e in our simplified example is the following sum:

$$P_E = p_{e1} + p_{e2}$$

$$= \int_0^{S_t} p_2(\hat{S})d\hat{S} + \int_{S_t}^{\infty} p_1(\hat{S})d\hat{S}$$

For any given pair of probability density functions p_1 and p_2 (e.g., Gaussian) there is a unique relationship between P_E, S_t and the moments of these functions. For example P_E is a function of S_t and $\sigma_1 \sigma_2$ (i.e.):

$$P_E = f[S_t, \sigma_1(\hat{S}), \sigma_2(\hat{S})]$$

where σ_1 and σ_2 are the standard deviations of p_1 and p_2 respectively.

In a typical signal detection problem a positive real value between zero and unity is assigned to P_E representing an acceptable level of error. Having previously determined the minimum acceptable ratio of signal to noise (r_{min}) and S_t, we can determine the maximum values for σ_1 and σ_2 from the allowed P_E. From these maximum standard deviations we can determine the required size of the sample space (N). The feasibility of the proposed signal detection scheme can be assessed from this value for N.

There are many other approaches for assessing feasibility based upon other sets of constraints. For example, the sample space size can be fixed by the sampling rate of the receiver, by the duration of the transmitter coherent light signal and by the computational capacity of the signal processing unit. However, we have chosen the above criteria for feasibility for illustrative purposes.

Understanding of this feasibility analysis is, perhaps, enhanced by specific numerical examples. In these examples we select a fixed r_{min}, a specific optical power level p_o wavelength λ bandwidth $\Delta\nu$ and detector bandwidth β_d. From these we compute the required sample size N from which $\hat{S}$ is estimated. We select $r_{min} = 0.1$ which is equivalent to -10 dB (power ratio) signal/noise. In the following examples the probability density functions p_1 and p_2 will be approximated by Gaussian functions for $\hat{S} \geq 0$. This approximation is reasonable provided the variances of p_1 and p_2 are sufficiently small or equivalently provided N is sufficiently large. In our first example the total received optical power is taken to be 10^{-10} watts at an average photon energy $<h\nu>$ of 10^{-20} Joule (i.e., near infrared). We assume that the sampling interval is $T = 10^{-8}$ sec which is equivalent to a nyquist sampling rate of a detector having a bandwidth of the order of 50 MHz. The average number of photons received in time T is given by

$$\langle n \rangle = \frac{Pt}{\langle n\nu \rangle}$$

$$= 100$$

In attempting to assess feasibility of this method with the following special cases we will refer to "large" and "small" value for $\langle n \rangle$. By such terms we imply large or small compared with $\langle n \rangle \cong 100$ photons/sampling interval. The implication of the magnitude for $\langle n \rangle$ involved in any particular case with regard to optical power level and bandwidth can be determined from a numerical estimate similar to the preceding example.

6.5 Special Case - Single Mode Fields

Although it will presently be shown that it is not a physically realistic example, single mode $(m = 1)$ fields are most easily interpreted. The variance $V(\hat{S})$ must be determined for our analysis. For signle mode fields the leading term in $V(\hat{S})$ for large $\langle n \rangle$ is

$$V(S) \xrightarrow[\langle n \rangle \to \infty]{} \frac{1}{N\langle n \rangle^2}\left[\frac{\langle n \rangle}{1 + r}\left(1 + \frac{\langle n \rangle}{1 + r}\right) + \left(\frac{r}{1 + r}\right)1 + \frac{2\langle n \rangle}{1 + r}\right]^2$$

$$V(\hat{S}) \xrightarrow[\langle n \rangle \to \infty]{} \frac{\langle n \rangle^2}{N(1+r)^2}$$

The standard deviation σ_1 and σ_2 are given by

$$\sigma_1 = \sqrt{V[\hat{S}(r)]} = \frac{\langle n \rangle}{1 + r}\frac{1}{\sqrt{N}}$$

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$$\sigma_2 = \sqrt{v[\hat{s}(0)]} = \frac{<n>}{\sqrt{N}}$$

Note that for $r \sim 0.1$ these two standard deviations are approximately equal, i.e.,

$$\sigma_1 \approx \sigma_2 \approx \sigma \frac{<n>}{\sqrt{N}}$$

The criterion for determining N can be understood with reference to Fig. 5 which is a sketch of the two probability density functions p_1 and p_2.
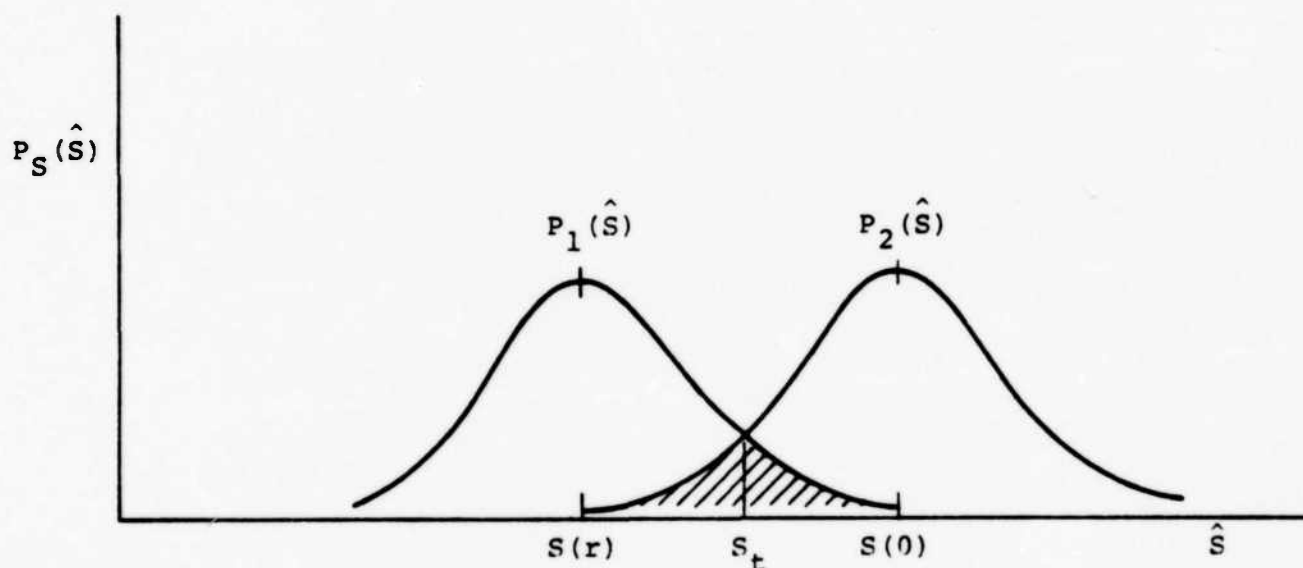

Figure 5

Recall that the total probability of error P_E is given by

$$P_E = \int_0^{S_t} p_2(\hat{S})d\hat{S} + \int_{S_t}^{\infty} p_1(\hat{S})d\hat{S}$$

$$\approx \ 2 \int_{S_t}^{\infty} p_1(\hat{S}) d\hat{S}$$

For the assumed Gaussian function P_E is given approximately by

$$P_E \ = \ 2\left[1 - \text{erf}\left(\frac{S_t - S(r)}{\sigma}\right)\right]$$

Taking the allowable P_E as 0.01 we obtain

$$S_t - S(r) \ = \ 4.8 \ \sigma \ = \ 2\left[S(0) - S(r)\right]$$

Earlier it was shown that $S(r)$ is given by

$$S(r) \ = \ \frac{<n>}{m} \frac{1 + 2r}{(1+r)^2}$$

For single mode fields we obtain

$$S(0) - S(r) \ = \ <n>\left[1 - \frac{1 + 2r}{(1+r)^2}\right]$$

$$= \ <n>\left(\frac{r}{1 + r}\right)^2$$

But we have shown for equal errors $(Pe_1 = Pe_2)$ that

$$S(0) - S(r) \Big|_{r_{min}} \ = \ 2(2.4) \ \sigma$$

$$\approx \ 4.8 \ \frac{<n>}{\sqrt{N}}$$

$$= \ <n>\left(\frac{r}{1 + r}\right)^2 \ \approx \ <n>r^2$$

Solving the above equation for the maximum N (r_{min}) yields

$$N = \frac{(4.8)^2}{r_{min}^4} \approx 25 \times 10^4$$

The feasibility of the proposed method is influenced by the memory capacity of the signal processing and the duration of the transmitted signal from which the N samples are taken. If the samples are taken in intervals $T = 10^{-8}$ sec, then the minimum signal duration for τ for all N samples must be

$$\tau = 10^{-8} N$$

$$= 2.5 \text{ msec}$$

In most practical cases a signal duration of 2.5 msec will not be limiting. However, a required memory capacity to store 250,000 data points is totally unrealistic. Thus it appears that the implementation of the proposed scheme using a computer or special purpose processor is not feasible.

On the other hand, a minimum signal/noise of 0.4 requires a memory capacity of only about $N \approx 1000$ samples which is clearly within the capability of a small digital processor.

Although our example feasibility study is not physically realistic, it does demonstrate a few of the fundamental

compromises between system complexity and performance of the proposed method. The sensitivity of the size of the sample space upon the required memory capacity is clearly demonstrated. The following numerical example further expands the understanding of the influence of system parameters upon performance.

6.6 Second Special Case Feasibility Study

Our second special case estimate of the feasibility of the proposed signal detection method considers multimode fields. In this second special case we once again assume relatively large number of received photons ($<n> \approx 100$) in the sampling interval T.

The number of modes m associated with the optical field can be computed from the configuration depicted in Fig. 6. It has been shown above that m is given by

$$m = \frac{SS^1 \Delta \nu}{R^2 \lambda^2 \beta_d}$$

We assume the following reasonable values for the parameters of the expressions for m:

$$S^1 = 0.1 \text{ m}^2$$
$$S = 10^{-4} \text{ m}^2$$
$$R = 1 \text{ Km}$$
$$\lambda = 1 \text{ } \mu m$$
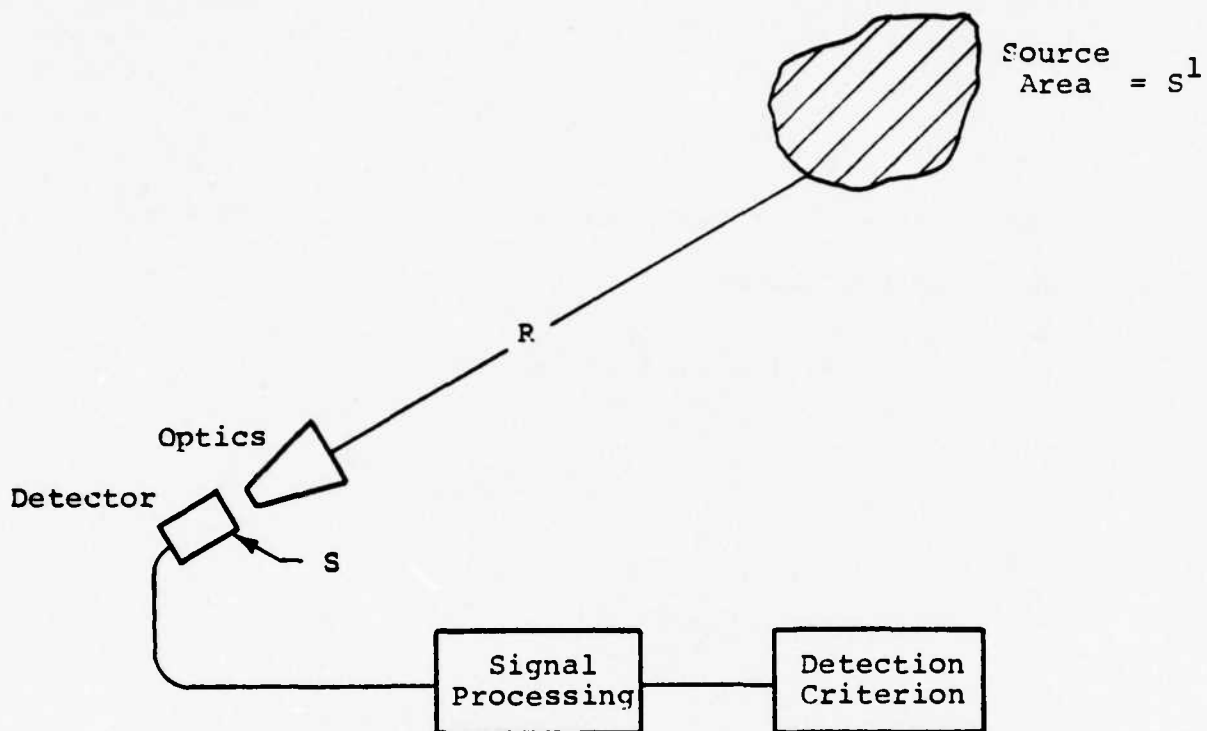$$\Delta \nu = 10^{-10}$$
$$\beta_d = 10^8$$

Figure 6

Substituting these numerical values into the equation for n yields:

$$m = \frac{10^{-4} \times 10^{-1} \times 10^2}{10^6 \times 10^{-12}}$$

For the above values of $\langle n \rangle$ and m the leading term for $V(\hat{S})$ is given by

$$V(\hat{S}) \approx \frac{\langle n \rangle^2}{N \langle n \rangle^2} = \frac{1}{N}$$

The standard deviation for p_1 and p_2 is given by

$$\sigma = \frac{1}{\sqrt{N}}$$

For a total $P_E = 0.01$ we have once again (assuming Gaussian functions):

$$S(0) - S(r) \approx 4.8 \, \sigma$$

$$= \frac{\langle n \rangle}{m}\left(\frac{r}{1+r}\right)^2 \approx \frac{5}{\sqrt{N}}$$

Solving for N yields:

$$N \approx 25\left(\frac{m}{\langle n \rangle}\right)^2\left(\frac{1+r}{r}\right)^4$$

Taking $r_{min} = 3$ and using the above determined values for m and $\langle n \rangle$ yields:

$$N \approx 7500$$

This is a reasonable sample size for a small general purpose computer or processor. If the memory capacity is increased to 16 K, then r_{min} is reduced to only

$$\min r \approx 2.3$$

Thus it appears that the proposed method is applicable for signal/noise of about 6 dB.

6.7 Third Special Case - Small $\langle n \rangle$

The third special case for which the feasibility of the proposed method is investigated is for small $\langle n \rangle$ (small compared with unity). The leading term in the expression for $V(\hat{S})$ in this case is:

$$V(S) \xrightarrow[\langle n \rangle \to 0]{} \frac{1}{\langle n \rangle^2 N} \{\langle n \rangle\}$$

$$= \frac{1}{\langle n \rangle N}$$

The standard deviation for this case is given by:

$$\sigma = \frac{1}{\sqrt{\langle n \rangle N}}$$

The criterion of a 0.01 total P_E is then:

$$S(0) - S(r) = \frac{\langle n \rangle}{m}\left(\frac{r}{1 + r}\right)^2$$

$$= 2\sigma$$

$$= \frac{2}{\sqrt{\langle n \rangle N}}$$

Solving for N yields:

$$N = \frac{4 m^2}{\langle n \rangle^2}\left(\frac{1 + r}{r}\right)^4$$

In interpreting this equation it is sensible to fix a reasonable value for N_{max} and to find the value for

$\langle n \rangle$. We have shown that for a typical configuration $m \approx 10^3$. A reasonable upper bound for N is of the order of 16×10^4. If we arbitrarily select $r = \frac{1}{2}$, we find

$$\langle n \rangle \approx (20)^{-\frac{1}{3}} = 0.3$$

This corresponds to an optical power of

$$P = \frac{\langle n \rangle}{T} h\nu$$

$$= \frac{10^{-20} \times 0.3}{10^{-8}}$$

$$= 0.3 \times 10^{-12} \text{ watts}$$

6.8 Fourth Special Case - Optimum $\langle n \rangle$

The next special case which we consider involves the selection of $\langle n \rangle$ for minimum variance in our estimate of signal $\hat{S}$. For a given optical power level we can choose $\langle n \rangle$ by selecting the sampling interval T (neglecting physical constraints).

$$\langle n \rangle = \frac{P}{\langle h\nu \rangle} T$$

It can be shown from an upper bound for $V(\hat{S})$ that there is an optimum $\langle n \rangle$ which minimizes the variance and hence the standard deviation σ.

As we have already shown the standard deviation represents the post signal processing noise. In the following

analysis we compute the average number of photons which are counted in the sampling interval T for which $V(\hat{S})$ is minimized. This simultaneously minimizes the standard deviation and noise. Thus the present special case can be considered the minimum noise case.

In Appendix C the following approximation is established for $V(\hat{S})$ for single mode fields:

$$V(S) \;<\; 2\,\frac{(1+2<n>)^2}{N}\left[6(1+r) \;+\; \frac{1}{<n>}\right]$$

Note that this function asymptotically approaches infinity for $<n> \to 0$ and for $<n> \to \infty$. Hence there must be a positive real $<n>$ which gives minimum $V(\hat{S})$ and hence minimum noise.

Letting $A = N\,V(\hat{S})$ and $a = <n>$ we can find the value of a which minimizes A by differentiating:

$$\frac{\partial A}{\partial a} \;=\; 0 \;=\; 4\left(1 + 2a\right)\left(6 + \frac{1}{a}\right) - \frac{(1+2a)^2}{a}$$

$$=\; 24\,a^2 + 2a - 1$$

$$=\; (6a - 1)(4a + 1)$$

The minimum for A occurs for

$$a \;=\; \frac{1}{6}$$

which yields

$$\min V(S) \;=\; \frac{A\left(\frac{1}{6}\right)}{N}$$

$$= \frac{\left(\frac{4}{3}\right)^2}{N} \quad (12)$$

$$= \frac{64}{3N}$$

It is further shown in Appendix C that for multimode fields the variance $V(\hat{S})$ is given approximately by

$$V(\hat{S}) \simeq \frac{1}{N}\left(1 + \frac{2<n>}{m}\right)^2\left[1 + \frac{5}{m} + 6r + \frac{1}{<n>}\right]$$

It is also shown there that this variance is minimized for

$$a = \frac{-\frac{2}{m} + \sqrt{\frac{4}{m^2} + 16\,bm}}{\frac{8b}{m}}$$

where

$$a = \text{value of } <n> \text{ which minimizes } V(\hat{S})$$

$$b = 1 + \frac{5}{m} + 6r$$

Earlier we have shown that a practical realization of the proposed signal detection method involves m of the order of 100 to 1000. For m greater than about 100 $a(m)$ is given approximately by

$$a(m) \approx \frac{1}{2}\sqrt{\frac{m}{1 + 6r}}$$

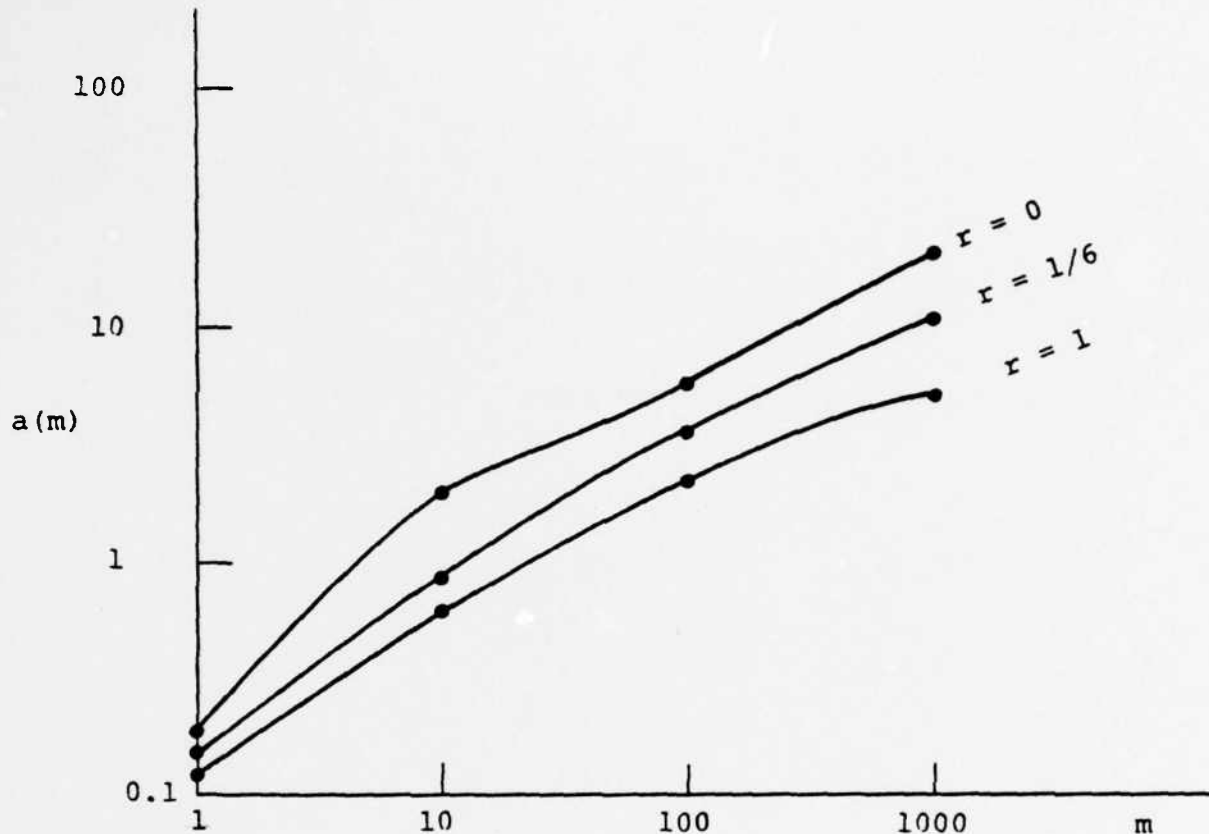Figure 7 is a graph of a(m) for a few representa-
tive values

Figure 7

Notice that as the signal/noise increases for any given m,
the required average photon count (<n>) decreases for the
minimum noise case. Notice further that in the normal
operating region (100 $\leq$ m $\leq$ 1000 and r $\leq$ 1) the average
number of photons in a sampling interval T is about 10.

For m $\geq$ 100 (the typical case) the variance $V(\hat{S})$
is given by

$$V(\hat{S}) \approx \frac{1 + 6r}{N}$$

The total probability of error P_E will be less than or equal to 0.01 provided

$$S(0) - S(r) = \frac{\langle n \rangle}{m}\left(\frac{r}{1 + r}\right)^2 \lesseqqgtr 5\sqrt{V(\hat{S})}$$

$$= 5\sqrt{\frac{1 + 6r}{N}}$$

The required sample space size for this case is given by

$$N = \frac{25(1+6r)}{\frac{1}{2}\sqrt{\frac{m}{1+6r}}} m\left(\frac{1 + r}{r}\right)^4$$

$$= 50(1 + 6r)^{\frac{3}{2}} m^{\frac{1}{2}}\left(\frac{1 + r}{r}\right)^4$$

If the memory capacity is limited to 16 K, then the minimum r is given by about

$$\min r \approx 1.4$$

This shows an improvement of about a factor of 2 over the minimum detectable signal/noise in the large $\langle n \rangle$ special case. Thus there are benefits to operating at optimum $\langle n \rangle$.

It is instructive to consider that the mechanism by which $\langle n \rangle$ is selected for any given received optical power P is made by selecting T. The relationship between $\langle n \rangle$,

P and T is given by

$$\langle n \rangle = \frac{PT}{h\langle \nu \rangle}$$

where $\langle \nu \rangle$ is the average optical frequency. This frequency is given by the center frequency of the received optical band. Typically this will be the band center of any optical preselector filter which precedes the detector.

For minimum noise and large m fields the average number of photons received in interval T is given approximately by

$$\langle n \rangle = \frac{PT}{h\langle \nu \rangle} \approx \frac{1}{2} \sqrt{\frac{m}{1 + 6r}}$$

We are most interested in detecting temporally coherent light at low signal levels and at low signal/noise (small r). Thus for any given power level P, the optimum choice of T is

$$T \approx \frac{h\langle \nu \rangle}{2P} \sqrt{m}$$

Of course, this result was obtained without regard to any constraints imposed by the detector bandwidth or the speed of the sampling electronics. In any practical implementation, operation at the minimum noise condition may not be feasible.

7.   SUMMARY OF RESULTS

This report has described studies of a method of detecting temporally coherent light in a noncoherent background.   It has been shown that the arrival rate statistics of photons are different for coherent and noncoherent light.   In this study the probability density function p(n) for the member n of photons which arrive in an interval T is a function of the ratio r of the average coherent to noncoherent photons.

In addition it has been shown that the various moments of p(n) depend upon r.   From this observation a post signal processing signal S has been defined:

$$S = \frac{V(n)}{\langle n \rangle} - 1$$

where

$$V(n) = \langle (n - \langle n \rangle)^2 \rangle$$

and where $\langle \ \rangle$ indicates expected value.   It has been shown that S is a function of r.   Moreover it is shown that in principle it is possible to distinguish the presence of temporally coherent light in a noncoherent background $(r > 0)$ from purely noncoherent light $(r = 0)$.

A perfect measurement of r would require a zero error measurement of S.   Unfortunately it is impossible

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to measure S without error. Rather, S can only be estimated from a finite sample space of measurements of the number of photons. This estimate of S is denoted $\hat{S}$:

$$S = \frac{b_N(n)}{a_N(n)} - 1$$

where

$$b_N = \frac{1}{(n-1)} \sum_{m=1}^{N} (n_m - a_N)^2$$

$$a_N = \frac{1}{N} \sum_{m=1}^{N} n_m$$

n_m = number of photons received in the mth sampling interval

In addition this study has shown that the error in this estimate of S as measured by its variance $V(\hat{S})$ is given approximately

$$V(S) \approx \frac{\mu_4 - 2\mu_2^2}{N\langle n\rangle^2}$$

where μ_4 and μ_2 are the fourth and second central moments of $p(n)$ respectively. In this approximation it has been assumed that N is large enough that

$$a_N \approx \lim_{N\to\infty} a_N = \langle n\rangle$$

The post signal processing noise N_o associated with a measurement of $\hat{S}$ is given by

$$N_o = \sqrt{V(\hat{S})} = \sigma(\hat{S})$$

The required size of the sample space N for detection of coherent light at a given r is determined for a given allowable error rate. The feasibility of this proposed signal conditioning is assessed by determining the physical parameters involved in the required sample space size. The limitations on the feasible sample space size include: the memory capacity and computation speed of the processor; the sampling rate, detector bandwidth, and coherent signal duration.

It has been shown that for small r unimodal fields the required sample space size is given by

$$N \approx \frac{25}{r^4}$$

If the memory capacity is to be limited to about 16 K, then the minimum r which can be detected is given by

$$\min r \approx 0.11$$

Next this study considered multimode fields in which $m \geq 100$. It was shown that relatively large $<n>$ (about 100) the required sample space size is given by

$$N \approx 25\left(\frac{m}{\langle n \rangle}\right)^2 \left(\frac{1+r}{r}\right)^4$$

As an illustrative example, it was shown that for min $r \approx 3$ the required sample space size is:

$$N \approx 7500$$

Although a memory capacity of 7500 is relatively small, increasing N to 16 K only reduces min r to about

$$\min r \Big|_{N = 16\,K} \approx 2.3$$

Thus it appears that the present method when applied to multimode fields having large $\langle n \rangle$ is limited to a signal/noise of about 7 dB.

On the other hand it was shown that there is an optimum $\langle n \rangle$ which minimizes noise. For $m \geq 1000$ that value of $\langle n \rangle$ is given by

$$n \approx \frac{1}{2}\sqrt{\frac{m}{1+6r}}$$

For this case it was shown that the required sample space is given by

$$N \approx 50\left(1+6r\right)^{\frac{3}{2}} m^{\frac{1}{2}} \left(\frac{1+r}{r}\right)^4$$

If the memory capacity is limited to about 16 K, then the minimum detectable r (for $P_E = 0.01$) is given by

$$\min r \approx 1.4$$

This is equivalent to about 1 dB signal/noise which is roughly a 6 dB improvement over the large <n> case. Thus it has been shown that there are benefits to operation at optimum <n>.

REFERENCES

1. L. Mandel, <u>Progress in Optics</u>, Vol. 11, North Holland, Amsterdam, p. 181.

2. Jan Peřina, <u>Coherence of Light</u>, Van Nostiano, N.Y., 1972, p. 283.

3. H. Cramér, <u>Matematical Methods of Statistics</u>, 13th printing, Princeton, 1974, p. 348.

4. Jan Peřina, loc cit, p. 140.

It is the purpose of this appendix to derive the recurrence relations for the moments of the probability density function $p(n)$:

$$p(n) = \frac{1}{(n+m-1)!}\left(1+\frac{m}{y}\right)^{-n}\left(1+\frac{y}{m}\right)^{-m} e^{-\left(\frac{x}{1+\frac{y}{m}}\right)} L_n^{m-1}(Z) \tag{A.1}$$

where

$$x = \langle n_c \rangle$$

$$y = \langle n_i \rangle$$

$$Z = -\frac{xm^2}{y(y+m)}$$

$$m = \text{number of modes}$$

$$L_n^{m-1}(Z) = \text{Laguerre polynomial}$$

For the purpose of establishing recurrence relations it is helpful to compute the following derivatives:

$$\frac{\partial P}{\partial x} = -\left(\frac{1}{1+\frac{y}{m}}\right)p(n) - \frac{m}{y\left(1+\frac{y}{m}\right)}\frac{1}{(n+m-1)!}\left(1+\frac{m}{y}\right)^{-n}\left(1+\frac{y}{m}\right)^{-m} e^{-\left(\frac{x}{1+\frac{y}{m}}\right)}L_n^{m-1}(Z) \tag{A.2}$$

$$\frac{\partial P}{\partial y} = \frac{1}{(n+m-1)!} L_n^{m-1}(Z)\left[-n\left(1+\frac{m}{y}\right)^{-n-1}\left(\frac{m}{y^2}\right) - m\left(1+\frac{y}{m}\right)^{-m-1}\left(\frac{1}{m}\right) + \frac{x}{\left(1+\frac{y}{m}\right)^2}\frac{1}{m}\right] \tag{A.3}$$

$$+ \frac{mx\left(1+\frac{2y}{m}\right)}{y\left(1+\frac{y}{m}\right)(n+m-1)!}\left(1+\frac{m}{y}\right)^{-n}\left(1+\frac{y}{m}\right)^{-m} e^{-\left(\frac{x}{1+\frac{y}{m}}\right)}L_n^{m-1\,'}(Z)$$

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For reasons which will become apparent we consider the following expression:

$$x\left(1+\frac{2y}{m}\right)\frac{\partial P}{\partial x} + y\left(1+\frac{y}{m}\right)\frac{\partial P}{\partial y} = -\frac{x\left(1+\frac{2y}{m}\right)}{1+\frac{y}{m}}\,p(n) \tag{A.4}$$

$$-\,mx\left(1+\frac{2y}{m}\right)\left(1+\frac{m}{y}\right)^{-n}\left(1+\frac{y}{m}\right)^{-m}e^{-\left(\frac{x}{1+\frac{y}{m}}\right)}L_n^{m-1}(Z)$$

$$+\left[\frac{nmy\left(1+\frac{y}{m}\right)}{y^2\,1+\frac{m}{y}} - y + \frac{xy}{m\left(1+\frac{y}{m}\right)}\right]p(n)\;+\;\alpha$$

$$\alpha = \frac{mx\left(1+\frac{2y}{m}\right)}{y\left(1+\frac{y}{m}\right)(n+m-1)!}\left(1+\frac{m}{y}\right)^{-n}\left(1+\frac{y}{m}\right)^{-m}e^{-\left(\frac{x}{1+\frac{y}{m}}\right)}L_n^{m-1}(Z)$$

$$=\left[-\frac{x\left(1+\frac{2y}{m}\right)}{1+\frac{y}{m}} + n - y + \frac{xy}{m\left(1+\frac{y}{m}\right)}\right]p(n) \tag{A.5}$$

$$=\left\{-x\left[\frac{1}{1+\frac{y}{m}} + \frac{y}{m\left(1+\frac{y}{m}\right)}\right] + n - y\right\}p(n) \tag{A.6}$$

$$=\left[n - (x+y)\right]p(n) \tag{A.7}$$

The recurrence relation can be obtained from Eq. A.7, by multiplying both sides by n^k and summing over all possible n.

$$\sum_{n=1}^{\infty} n^k \left\{ x \left(1 + \frac{2y}{m}\right) \frac{\partial p}{\partial x} + y \left(1 + \frac{y}{m}\right) \frac{\partial p}{\partial y} \right\} = \sum_{n=1}^{\infty} \left[n - (x+y) \right] p(n) \tag{A.8}$$

$$= x \left(1 + \frac{2y}{m}\right) \frac{\partial \langle n^k \rangle}{\partial x} + y \left(1 + \frac{y}{m}\right) \frac{\partial \langle n^k \rangle}{\partial y} = \langle n^{k+1} \rangle - (x+y) \langle n^k \rangle$$

This can be rewritten in a more easily recognized form:

$$\langle n^{k+1} \rangle = \langle n^k \rangle (x+y) + x \left(1 + \frac{2y}{m}\right) \frac{\partial \langle n^k \rangle}{\partial x} + y \left(1 + \frac{y}{m}\right) \frac{\partial \langle n^k \rangle}{\partial y} \tag{A.9}$$

Equation A.9 is the recurrence relation which we are seeking.

APPENDIX B

It is the purpose of this appendix to derive an expression for $V(\hat{S})$. This represents the mean squared noise associated with the proposed signal processing method. In the text it is shown that

$$V(\hat{S}) \approx \frac{\mu_4 - 2\mu_2^2}{N\langle n\rangle^2} \tag{B.1}$$

where

$$\mu_4 = \langle (n - \langle n\rangle)^4 \rangle$$

$$\mu_2 = \langle (n - \langle n\rangle)^2 \rangle$$

These last central moments of $p(n)$ can be computed from the first four moments of $p(n)$ using the recurrence relation of Appendix A:

$$\mu_4 = \langle n^4 - 4n^3\langle n\rangle + 6n^2\langle n\rangle^2 - 4n\langle n\rangle^3 + \langle n\rangle^4 \rangle \tag{B.2}$$

$$= \langle n^4\rangle - 4\langle n\rangle\langle n^3\rangle + 6\langle n\rangle^2\langle n^2\rangle - 3\langle n\rangle^4$$

and

$$\mu_2 = \langle n^2\rangle - \langle n\rangle^2$$

In Appendix A it is shown that

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$$\langle n^{k+1} \rangle = (x+y)\langle n^k \rangle + x\left(1+\frac{2y}{m}\right)\frac{\partial \langle n^k \rangle}{\partial x} \tag{B.3}$$

$$+ \ y\left(1+\frac{y}{m}\right)\frac{\partial \langle n^k \rangle}{\partial y}$$

where

$$\langle n_c \rangle = x$$

$$\langle n_i \rangle = y$$

$$\langle n_c + n_i \rangle = x + y = \langle n \rangle$$

$$m = \text{number of modes}$$

The required moments can be found in ascending order from Eq. B.3. First we compute $\langle n^2 \rangle$:

$$n^2 = (x+y)^2 + x\left(1+\frac{2y}{m}\right) + y\left(1+\frac{y}{m}\right) \tag{B.4}$$

For the purpose of computing $\langle n^3 \rangle$ we find the derivatives:

$$\frac{\partial \langle n^2 \rangle}{\partial x} = 2(x+y) + \left(1+\frac{2y}{m}\right) \tag{B.5}$$

$$\frac{\partial \langle n^2 \rangle}{\partial y} = 2(x+y) + 2\frac{x}{m} + \left(1+\frac{2y}{m}\right) \tag{B.6}$$

We then obtain

$$\langle n^3 \rangle = (x+y)^3 + \left[x\left(1+\frac{2y}{m}\right) \right. \tag{B.7}$$

$$\left. + \ y\left(1+\frac{y}{m}\right)\right]\left[3(x+y) + \left(1+\frac{2y}{m}\right)\right] + \frac{2xy}{m}\left(1+\frac{y}{m}\right)$$

The derivatives of Eq. B.7 are needed to find $\langle n^4 \rangle$:

$$\frac{\partial \langle n^3 \rangle}{\partial x} = 3(x+y)^2 + 3\left[x\left(1+\frac{2y}{m}\right) + y\left(1+\frac{y}{m}\right)\right] \tag{B.8}$$

$$+ \left(1+\frac{2y}{m}\right)\left[3(x+y) + \left(1+\frac{2y}{m}\right)\right]$$

$$+ 2\frac{y}{m}\left(1+\frac{y}{m}\right)$$

$$\frac{\partial \langle n^3 \rangle}{\partial y} = 3(x+y)^2 + 3\left[x\left(1+\frac{2y}{m}\right) + y\left(1+\frac{y}{m}\right)\right] \tag{B.9}$$

$$+ \frac{2}{m}\left[x\left(1+\frac{2y}{m}\right) + y\left(1+\frac{y}{m}\right)\right]$$

$$+ \left[\frac{2x}{m} + 1 + \frac{2y}{m}\right]\left[3(x+y) + \left(1+\frac{2y}{m}\right)\right]$$

$$+ \frac{2x}{m}\left(1+\frac{2y}{m}\right)$$

From B.1, B.8, and B.9 we obtain

$$\langle n^4 \rangle = 3(x+y)^2\left[x\left(1+\frac{2y}{m}\right) + y\left(1+\frac{y}{m}\right)\right] + 3\left[x\left(1+\frac{2y}{m}\right) + y\left(1+\frac{2y}{m}\right)^2\right.$$

$$+ 3(x+y)\left(1+\frac{2y}{m}\right)\left[x\left(1+\frac{2y}{m}\right) + y\left(1+\frac{y}{m}\right)\right] + \left(1+\frac{2y}{m}\right)^2\left[y\left(1+\frac{y}{m}\right)\right.$$

$$+ x\left(1+\frac{2y}{m}\right)\bigg] + 6xy\left(1+\frac{y}{m}\right)\left[x + y + 1 + \frac{2y}{m}\right]$$

$$+ \frac{2y}{m}\left(1+\frac{y}{m}\right)\left[x\left(1+\frac{2y}{m}\right) + y\left(1+\frac{y}{m}\right)\right]$$

$$+ (x+y)^4 + 3(x+y)^2\left[x\left(1+\frac{2y}{m}\right) + y\left(1+\frac{y}{m}\right)\right] \tag{B.10}$$

$$+\ 2\ \frac{xv}{m}\left(1+\frac{y}{m}\right)(x+y)\ +\ (x+y)\left(1+\frac{2y}{m}\right)\left[x\left(1+\frac{2y}{m}\right)\ +\ y\left(1+\frac{y}{m}\right)\right]$$

We obtain μ_4 by substituting Eq. B.10, B.7 and B.4 into Eq. B.2:

$$\mu_4\ =\ 3\left[x\left(1+\frac{2v}{m}\right)\ +\ y\left(1+\frac{y}{m}\right)^2\right.$$

$$+\ \left[y\left(1+\frac{y}{m}\right)\ +\ x\left(1+\frac{2v}{m}\right)\right]\left[\left(1+\frac{2v}{m}\right)^2\ +\ \frac{2y}{m}\left(1+\frac{y}{m}\right)\right]$$

$$\left.+\ 6xy\left(1+\frac{y}{m}\right)\left(1+\frac{2v}{m}\right)\right] \tag{B.11}$$

Similarly we find μ_2:

$$\mu_2\ =\ y\left(1+\frac{v}{m}\right)\ +\ x\left(1+\frac{2v}{m}\right) \tag{B.12}$$

The post signal processing noise involves the binomial $\mu_4 - 2\mu_2^2$ which is given by

$$\mu_4 - 2\mu_2^2\ =\ \left[x\left(1+\frac{2v}{m}\right)\ +\ y\left(1+\frac{v}{m}\right)\right]^2$$

$$+\ \left[y\left(1+\frac{y}{m}\right)\ +\ x\left(1+\frac{2v}{m}\right)\right]\left[\left(1+\frac{2y}{m}\right)^2\ +\ \frac{2y}{m}\left(1+\frac{y}{m}\right)\right]$$

$$+\ 6xy\left(1+\frac{y}{m}\right)\left(1+\frac{2v}{m}\right) \tag{B.13}$$

The noise associated with the proposed signal detection method is most easily interpreted if we eliminate x and y from B.13 in favor of system variables. We define the input signal to noise ratio r as

$$r = \frac{x}{y} = \frac{\langle n_c \rangle}{\langle n_i \rangle}$$

Noting that $\langle n \rangle = x + y$ we find

$$x = \frac{r\langle n \rangle}{1 + r}$$

$$y = \frac{\langle n \rangle}{1 + r}$$

Substituting these expressions for x and y into Eq. B.13 we can obtain $v(\hat{S})$ from Eq. B.1:

$$V(\hat{S}) = \frac{1}{N\langle n \rangle^2} \left\{ \left[\frac{r\langle n \rangle}{1 + r}\left(1 + \frac{2\langle n \rangle}{m(1+r)}\right) \right. \right.$$

$$+ \left. \frac{\langle n \rangle}{1 + r}\left(1 + \frac{\langle n \rangle}{m(1+r)}\right) \right]\left[\frac{r\langle n \rangle}{1 + r}\left(1 + \frac{2\langle n \rangle}{m(1+r)}\right) + \frac{3\langle n \rangle}{m(1+r)}\left(1 + \frac{\langle n \rangle}{m(1+r)}\right) \right.$$

$$+ \left. \left(1 + \frac{2\langle n \rangle}{m(1+r)}\right)^2 \right] + \frac{6r\langle n \rangle^2}{(1+r)^2}\left(1 + \frac{\langle n \rangle}{m(1+r)}\right)\left(1 + \frac{2\langle n \rangle}{m(1+r)}\right) \right\}$$

The post signal processing noise N_o is given by

$$N_o = \sqrt{V(\hat{S})}$$

APPENDIX C

It is the purpose of this appendix to show that there is an optimum average number of received photon $<n>$ in the sampling interval which minimize the post signal processing noise N_o. It is shown in the text that

$$N_o = \sqrt{V(\hat{S})}$$

and as $V(\hat{S})$ is monotomic nondecreasing function of $V(\hat{S}) \geq 0$, then the value of $<n>$ which minimizes $V(\hat{S})$ also minimizes N_o.

The expression derived for $V(\hat{S})$ in Appendix B is complicated and somewhat difficult to interpret. On the other hand, this expression has an upper bound which very closely approximates $V(\hat{S})$ for a large number of modes m. Recall that any practical implementation of this method involves m of the order of 10^2 to 10^3. Thus it is meaningful to consider the upper bound for the purpose of finding optimum $<n>$.

In Appendix B it is shown that $V(\hat{S})$ is given by

$$V(\hat{S}) = \frac{1}{N<n>^2} \left[\frac{r<n>}{1+r}\left(1+\frac{2<n>}{m(1+r)}\right) \right. \tag{C.1}$$

$$\left. + \frac{<n>}{1+r}\left(1+\frac{<n>}{m(1+r)}\right)\right]\left[\frac{r<n>}{1+r}\left(1+\frac{2<n>}{m(1+r)}\right) + \frac{3<n>}{m(1+r)}\left(1+\frac{<n>}{m(1+r)}\right)\right.$$

$$+ \left(1+\frac{2<n>}{m(1+r)}\right)^2\right] + \frac{6r<n>^2}{(1+r)^2}\left(1+\frac{<n>}{m(1+r)}\right)\left(1+\frac{2<n>}{m(1+r)}\right)$$

The following inequality can be written for Eq. C.1:

$$V(\hat{S}) \; < \; \frac{1}{N<n>^2}\left\{\left[<n>\left(1+\frac{2<n>}{m}\right)\right.\right.$$

$$+ \; <n>\left(1+\frac{<n>}{m}\right)\right]\left[<n>\left(1+\frac{2<n>}{m}\right) \; + \; 3\;\frac{<n>}{m}\left(1+\frac{<n>}{m}\right)\right.$$

$$+ \; \left(1+\frac{2<n>}{m}\right)^2\right]\right\} + 6r<n>^2\left(1+\frac{<n>}{m}\right)\left(1+\frac{2<n>}{m}\right) \tag{C.2}$$

$$< \; \frac{\left(1+2\;\frac{<n>}{m}\right)^2}{N<n>^2}\left\{2<n>\left[<n>\left(1+\frac{3}{m}\right) \; + \; 1 \; + \; \frac{2<n>}{m}\right]\right\}+ \frac{6r}{N}$$

where we have used the inequality

$$1 \; + \; \frac{<n>}{m} \; < \; 1 \; + \; \frac{2<n>}{m} \tag{C.3}$$

For convenience we define the upper bound for $V(\hat{S})$ as A/N:

$$V(\hat{S}) \; < \; \frac{A}{N} \tag{C.4}$$

where

$$A \; = \; 2\left(1 \; + \; \frac{2<n>}{m}\right)^2\left(1 \; + \; \frac{5}{m} \; + \; 6r \; + \; \frac{1}{<n>}\right) \tag{C.5}$$

Letting the value of $<n>$ which minimizes A be denoted
a we obtain

$$A = \left(1 + \frac{2a}{m}\right)^2 \left(1 + 6r + \frac{5}{m} + \frac{1}{a}\right) \tag{C.6}$$

The value of which minimizes A can be found by differentiating;

$$\frac{\partial A}{\partial a} = 0 = \frac{4}{m}\left(1 + \frac{2a}{m}\right)\left(1 + 6r + \frac{5}{m} + \frac{1}{a}\right) \tag{C.7}$$

$$- \frac{\left(1 + \frac{2a}{m}\right)^2}{a^2}$$

$$= \frac{4a^2}{m}\left(b + \frac{1}{a}\right) - \left(1 + 2\,\frac{a}{m}\right) \tag{C.8}$$

where

$$b = 1 + \frac{5}{m} + 6r \tag{C.9}$$

Solving for a yields

$$a = \frac{-1 + \sqrt{1 + 4bm}}{4b} \tag{C.10}$$

For large m (the most important practical case) $a(m)$ approaches

$$a(m) \longrightarrow \frac{1}{2}\sqrt{\frac{m}{1 + 6r}}$$

$$m \longrightarrow \infty$$

The result is useful for interpreting the optimum value of $<n>$.

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